Student-Led eLearning Modules: ModuleX: Quadrupole Ion Traps

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1 Introduction

This report is one of student-led online learning modules written by physics research students and made available by SEPnet. The aim of these modules is to support peers in the earlier stages of their research through transfer of information from the more experienced research student. This particular module is intended for students carrying out research in the field of ion trapping. In many cases, an early experimental physics research student is expected to dive straight into an ongoing experiment and learn on the go instead of forming a solid foundation at the start of their research. It is more so the case in the field of ion trapping where it is common for students to inherit the experimental set up of predecessors. Having only got a conceptual knowledge of the experiments and due to the lack of undergraduate taught material of this specialised field the student has to rely on the exchange of information from predecessors and their theses, from supervisors and from may technical papers from which the student has to extract the relevant information. Whilst the process of seiving through many materials can serve to broaden one's knowledge, it can be overwhelming and consumes a lot of time that is better spent on the research problem at hand.

I have assembled here material and sources that I believe will give a junior ion trapper an introduction to how Quadrupole Ion Traps work as well as a comprehensive theorectical treatement of the trapping potential. The module starts by introducing ion traps and their uses. A fully functional ion trap inevitably requires many working components; all are briefly discussed here including the use of pumping systems to get a trap into vacuum and electrical resonators to produce impedance matched high voltages. A full theoretical derivation of the trapping potential and the solutions to the Mathieu equations are provided. Pointers and useful tips are provided at the end.

2 Paul Ion Traps

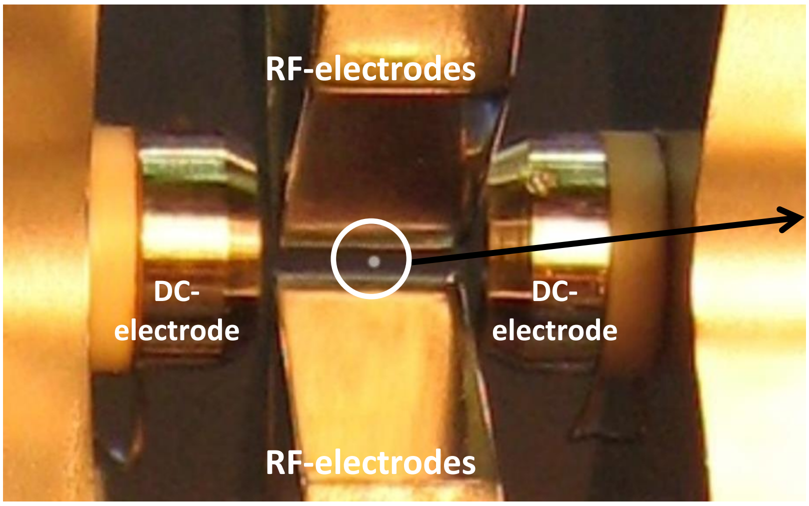
> Looking around oneself with a cynical, half-squinted eye one might not be immediately impressed by an endeavour whose pirincipal aim is to confine and atom to a space around 100 times larger than the Bohr radius - it being clear in these enlightened times that atoms in the solid state are actually very well localised - however, further consideration of the details may convince one otherwise. [1]

Ion Traps have made key contributions to many fields of physics: precision spectroscopy, atomic clocks, quantum state engineering,... The field of ion trapping has been steadily growing since the inventions of the Penning trap in 1936 [2] and the quadrupole mass filter in 1953 [3]. In 1989, the Nobel Prize in Physics was half jointly awarded to Hans. G. Dehmelt and Wolfgang Paul *"for the development of the ion trap technique"*. And more recently in 2012, the Physics Nobel prize went to David J. Wineland, jointly with Serge Haroche, for *"ground-breaking experimental methods that enable measuring and manipulation of individual quantum systems"*. David's research had a specific focus on laser cooling of trapped ions and use of trapped ions for the implementation of quantum information, control and computing.

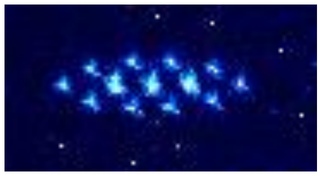
There are two main types of ion traps: Penning Traps and Quadrupole Ion Traps. Penning traps use DC electric fields combined with magnetic fields to trap ions. Quadupole Ion traps (also refered to as Paul or rf traps) use radiofrequency (rf) electric fields and DC electric fields to perform the same task. For the purposes of performing quantum state controls (for ion-trap based quatum computers for example) Paul traps present a lot less complications than Penning traps. In this module, we focus only on Paul Traps.

Paul traps themselves can take on different forms. The three main forms are Colinear Traps, Endcap Traps and Surface Traps. Each type has its own benefit and drawbacks. But all forms use the same principle to trap ions. We look next at examples of Linear Ion traps and Endcap Style ion traps. The region of electric pseudo-potential minimum in endcap style ion traps is defined by a point, as such, these traps are predominantly used to trap single ions. On the other hand, colinear traps have a whole axis where the pseudo-potential has a minimum. These are thus used for trapping strings of ions or large *Coulomb* crystals.

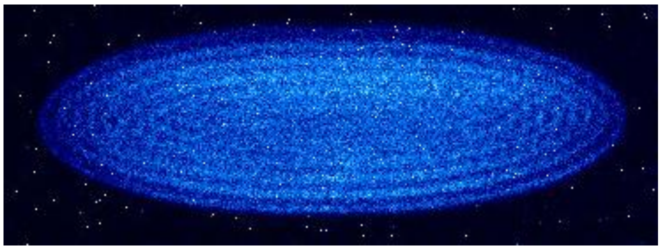
Example of a Colinear Traps



Ions as Viewed using a CCD Camera

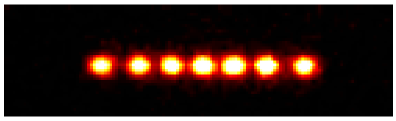


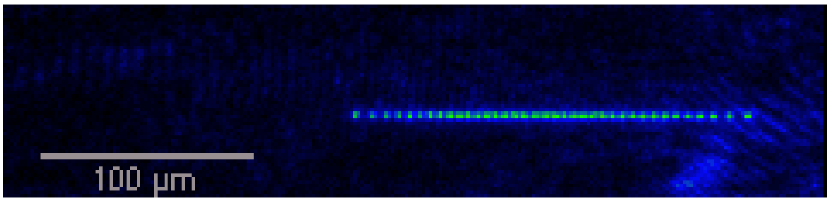
Coulomb Crystal of Ions



Larger Coulomb Crystal

String of Ions

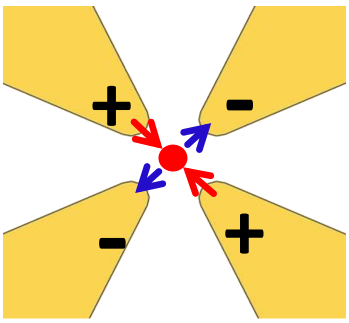




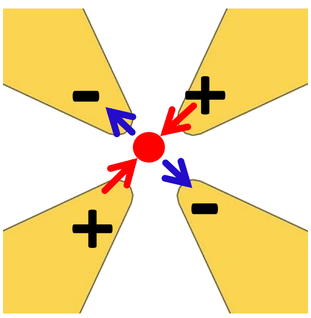
Ions can be easily manipulated by electric fields

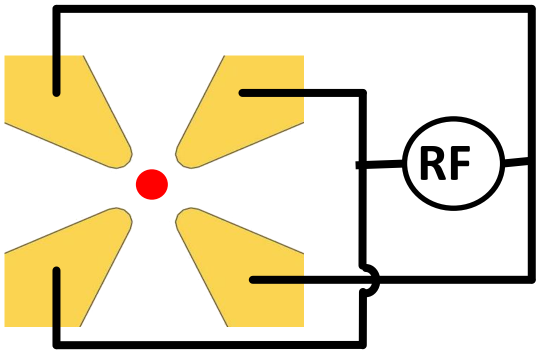
Working Principle

Step 1



Step 2



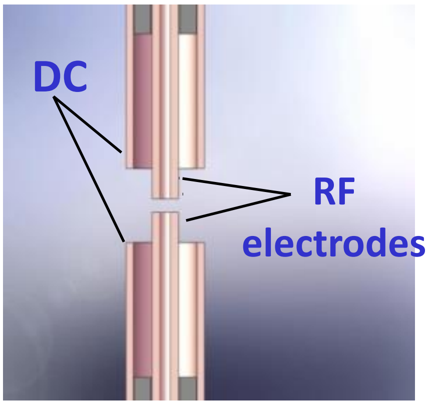


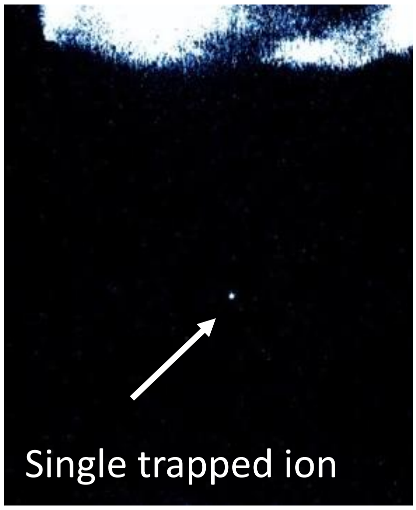
In step 1, opposite voltages are applied to adjacent electrodes. The ion (red dot) ‘sees’ the electric fields shown in arrows. In step 2, the voltages are reversed.

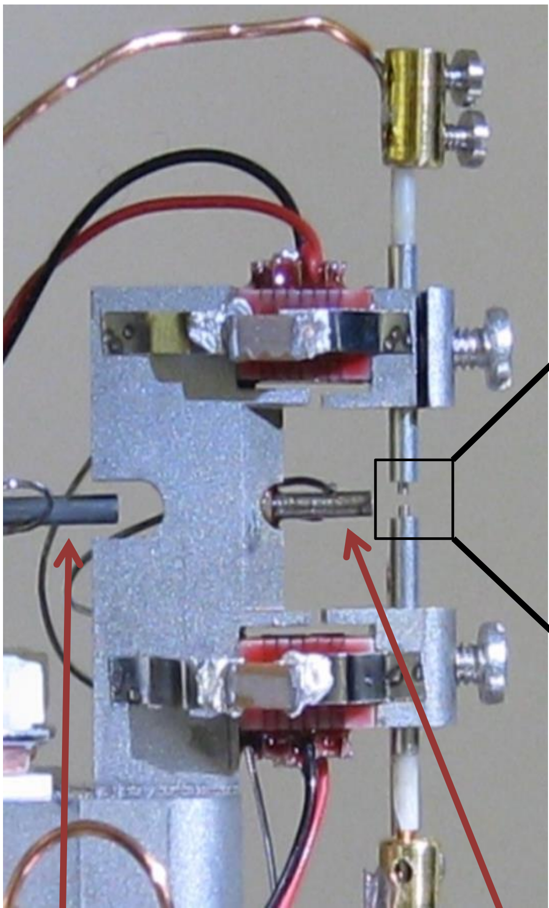
Alternate between step 1 and step 2 to keep ion localised.

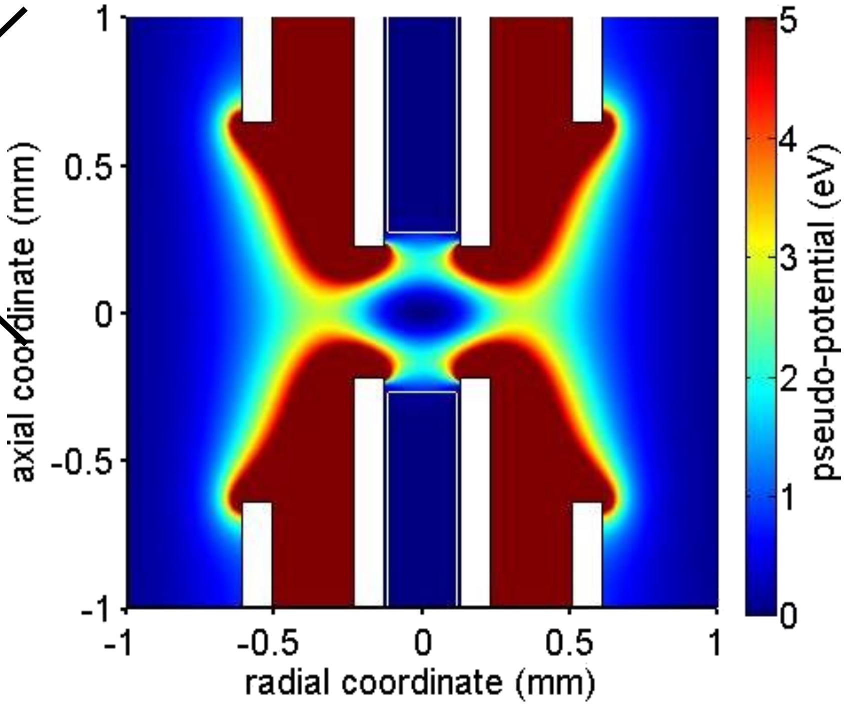
Example of a Endcap Traps

Useful for trapping single ions









Trap Depth: This is a simulation of the trapping potential for the trap shown when an 20 MHz ac voltage of amplitude 200 V is applied to the rf electrodes. The ion will be localised in the centre.

Atomic Oven: Essentially a metallic cylinder blocked at the rear end. It is stuffed with a ‘powder’ form of the atomic species of interest for trapping. The oven is resistively heated to excite atoms and generate an atomic beam toward the trap.

Collimator: This collimates the atomic beam from the over to flow between the trapping rf electrodes.

Trapping: Once there’s an atomic beam passing between the rf electrodes, it suffices to ionise the atoms by means of an ionisation laser at which point the ions will ‘see’ an effective potential similar to that shown in the plot.

3 The Trapping Potential and Mathieu's Equations [4]

3.1 The Trapping Potential

Earnshaw's Theorem [5] states that it is impossible to confine a charged particle in 3D using electrostatic forces alone. To prove this we consider a positively charged particle at the centre of a conducting spherical shell. If we apply a negative DC voltage with the aim of attracting the positively charged particle equally on all sides, it is inevitable that the particle will fall to the closest part of the shell. Now consider the case where we apply a positive DC to the spherical shell with the aim of repelling the particle in all directions. In this set up all field lines point inwards. If we want the the particle to be in a stable equilibrium about the centre, then the force field lines should point to this equilibrium position. If all force fields point inwards, then the divergence of the field at the centre must be negative. This is however impossible according to Gauss' law in free space which states that the divergence of the electric field is zero, $\nabla \cdot \mathbf{E} = 0$.

Having now established that we cannot confine an ion using electrostatic fields alone, we look for a non-static potential, ϕ, which allows the stable confinement of an ion in a region in space. Since $\nabla \cdot \mathbf{E} = \nabla \cdot (-\nabla\phi)$, it suffices to solve Laplace's equation, $\nabla^2\phi = 0$, to find a potential which can be used in ion trapping.

The equation which is the lowest order expansion of the potential which obeys Laplace's equation above in 3D Cartesian space (x, y, z) is of the form

$$\phi = k(\alpha x^2 + \beta y^2 + \gamma z^2) \tag{1}$$

with constant k and condition $\alpha + \beta + \gamma = 0$.

We note from the condition above that not all coefficients α, β and γ can simultaneously be positive (nor negative). In physical terms, at any given time, there could be a positive confining potential in two of the three dimensions but there must also be an *anti-trapping* potential in the third dimension.

We set $\alpha = \beta = 1$. Then, $\gamma = -2$. This choice gives a cylindrically symmetric potential:

$$\phi = k(r^2 - 2z^2) \tag{2}$$

where $r^2 = x^2 + y^2$ is the radial distance in the $x - y$ plane from the origin.

Ion traps producing equipotentials of the above equation are made of a pair of hyperboloid endcap electrodes of revolution about the z-axis and a ring electrode around the z-axis, also with a hyperbolic cross section. These traps are commonly referred to as Paul traps after Wolfgang Paul who shared the Physics Nobel Prize in 1989 for this invention.

Paul traps can be operated in different ways. An AC potential can be applied to both the endcap electrodes and the ring electrode but with a phase difference of 180°. Alternatively, the potential ϕ can be applied to the endcap electrodes whilst grounding the ring electrode (or vice-versa). Clearly, with the latter option, we need not concern ourself with relative phase optimisation between the endcaps and the ring. We choose to ground the ring electrodes and apply a potential ϕ_0 to both endcaps (with no phase difference),

$$\phi_0 = U_{dc} + V_{ac}\cos(\omega t) \tag{3}$$

where U_{dc} is the DC potential of applied to the endcap electrodes in addition to the radio frequency (RF) voltage with angular frequency ω with peak amplitude V_{ac}.

When the endcap and ring hyperbolas share the same asymptotes, consideration of the boundary conditions yields

$$\phi_0 = \phi(0, z_0, t) - \phi(r_0, 0, t) \tag{4}$$

$$= k(-2z_0^2 - r_0^2). \tag{5}$$

Thus, our trapping potential becomes

$$\phi = \frac{-\phi_0}{2z_0^2 + r_0^2}(r^2 - 2z^2) \tag{6}$$

The equations of motion of a particle of mass m and positive charge e in the Paul trap are

$$m\ddot{r} = eE_r = -e\frac{\partial \phi}{\partial r} \tag{7}$$

$$m\ddot{z} = eE_z = -e\frac{\partial \phi}{\partial z}. \tag{8}$$

Thus the equations of motion in the radial plane becomes

$$m\ddot{r} = \frac{2e}{r_0^2 + 2z_0^2}(U_{dc} + V_{ac}\cos(\omega t))r \tag{9}$$

The above is in the form of the Mathieu equation

$$\frac{d^2x}{d\tau} + (a - 2q\cos(2\tau)) = 0 \tag{10}$$

Going into dimensionless units by setting $\tau = \omega t/2$, we have $\mathrm{d}t^2 = 4\ \mathrm{d}\tau^2/\omega^2$. Then,

$$\frac{\mathrm{d}^2 r}{\mathrm{d}\tau^2} + \frac{8e}{m\omega^2(r_0^2 + 2z_0^2)}(-U_{dc} - V_{ac}\cos(2\tau))r = 0. \tag{11}$$

And thus the a and q parameters of the Mathieu equation for the radial equation of motion are respectively,

$$a_r = \frac{-8eU_{dc}}{m\omega^2(r_0^2 + 2z_0^2)}, \tag{12}$$

$$q_r = \frac{4eV_{ac}}{m\omega^2(r_0^2 + 2z_0^2)}. \tag{13}$$

Similarly, or by inspection of equation 6, we find the a and q parameters for the axial equations to be respectively

$$a_z = -2a_r = \frac{16eU_{dc}}{m\omega^2(r_0^2 + 2z_0^2)}, \tag{14}$$

$$q_z = -2q_r = \frac{-8eV_{ac}}{m\omega^2(r_0^2 + 2z_0^2)}. \tag{15}$$

3.2 Solutions to the Mathieu Equation and the Stability Criteria [4]

It takes some 5 theorems and 5 corollaries to find the solutions to the Mathieu equation (10). Later, I explicitly derive the solutions to the Mathieu equation. The object of this section is to impose physical limitations to extract the stable solutions from the general solutions. The general solution were found to be classified in two cases.

Solution type 1: The periodicity exponent of the two solutions are different and thus form two independent solutions. They have the relation $\mu_1 = -\mu_2$. Writing $\mu = \mu_1$, we have the even, r_+, and odd, r_-, solutions as follows.

$$r_\pm(t) = e^{\mu\tau}\phi(\tau) \pm e^{-\mu\tau}\phi(-\tau). \tag{16}$$

Solution type 2: The periodicity exponents are identical. We also have the condition $\mu_1 + \mu_2 = 2ni$ where n is an integer and i is the imaginary unit. This condition results in one of the solutions being periodic in which case the second solution takes a non-periodic and non-pseudoperiodic form. For the case where $r_1(t)$ is the periodic solution, the second solution becomes

$$r_2(t) = \pi r_2(\pi) \cdot t \cdot r_1(t) + u(t) \tag{17}$$

where $u(t)$ is a periodic function with period π.

Stability

Stability means that the position of the particle being trapped is bounded at all times, i.e. we do not lose the particle. We can straightforwardly see that *solution type 2* is unstable. Even though one of its solutions is periodic, the second solution has a term proportional to t. Therefore as $t \to \infty$, $r \to \infty$. We get this type of solution when $\mu_1 = \mu_2$. But since $\mu_1 + \mu_2 = 2ni$, the condition to get this type of solution becomes to have integer β where $\mu = i\beta$. Hence to avoid unstable solutions one condition is $\mu \neq ni$.

For *solution type 1* where $\mu_1 = -\mu_2$, the relation $\mu_1 + \mu_2 = 2ni$ is satisfied by any real entries α and β in $\mu = \alpha + i\beta$. However consideration of stability solutions means that μ must not have a real component as otherwise, both solutions will have terms which will grow exponentially in time. Thus our second condition for stability is to have μ as purely imaginary.

Altogether, the condition for stability is to have $\mu = i\beta$ where β is a non-integer.

For each stable solution, i.e. for each $\mu = i\beta$ where β is a non-integer, there are a and q parameters which would satisfy the Mathieu Equation. Integer values of β mark

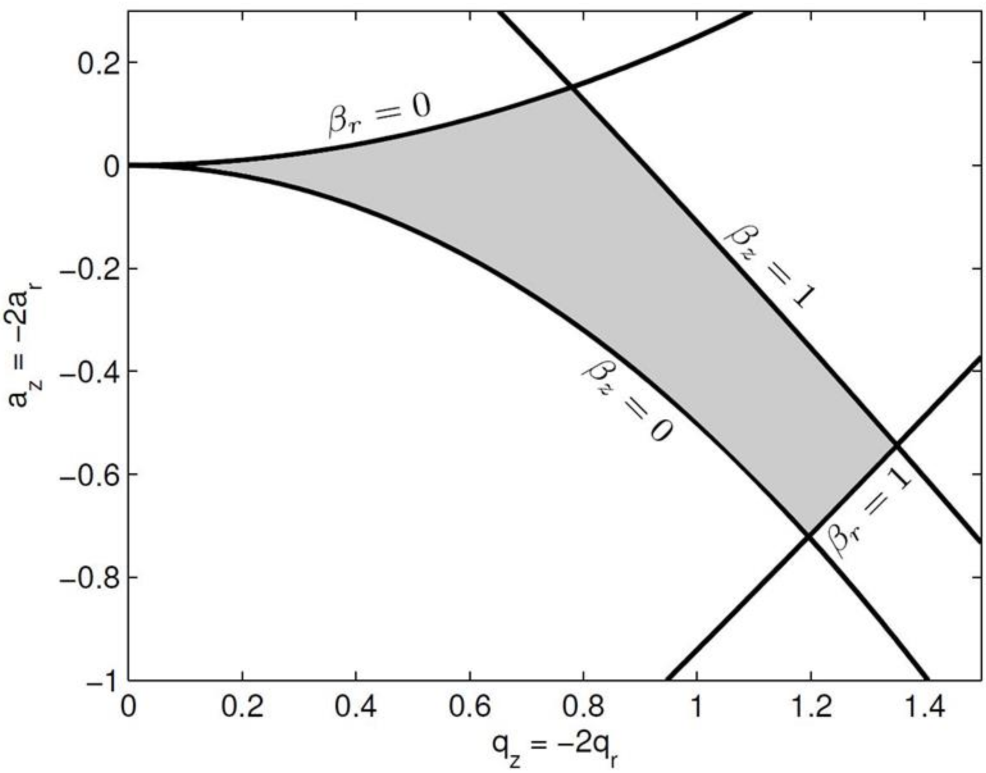


Figure 1: $\beta_{z,r} = 0, 1$ mark the boundaries of the stability region of interest. Our ion can be stably confined if our a and q parameters are manipulated to fall within the shaded region.

the boundaries of stable and unstable solutions. Recall

$$a_z = -2a_r \propto U_{dc} \text{ and} \tag{18}$$

$$q_z = -2q_r \propto V_{ac}. \tag{19}$$

Typically, a Paul trap has a fixed RF drive frequency ω and the DC voltage is varied to stabilise the system. The values of a_z (a_r) which give stable solutions for any given q_z (q_r) can iteratively be derived. The stability diagram in a-q space is plotted in figure 1. This can be used to find the possible Rf and DC voltage amplitudes required to succesfully trap ions.

4 Quiz: Solving the Mathieu Equation

The reader should now attempt to prove the following 6 theorems and 5 corollaries involved in solving the Mathieu Equation,

$$\frac{d^2w}{dz^2} + (a - 2qcos(2z)) = 0, \tag{20}$$

and produce the same conclusion as given at the end of this section. Note there is a cheat sheet in the next section if you need some tips.

Theorem 1. *If $w_1(z)$ is a solution, then so is $w_1(n\pi + z)$*

Theorem 2. *$\exists$ one odd and one even solution*

Theorem 3.

i) $w_1(z)$ is even and $w_2(z)$ is odd

ii) $w_1(0) = w_2'(0) = 1; w_1'(0) = w_2(0) = 0$

iii) $w_1(z \pm \pi) = w_1(\pi)w_1(z) \pm w_1'(\pi)w_2(z)$

iv) $w_2(z \pm \pi) = \pm w_2(\pi)w_1(z) + w_2'(\pi)w_2(z)$

v) $w_1(z)w_2'(z) - w_1'(z)w_2(z) = 1$

vi) $w_1(\pi) = w_2'(\pi)$

Theorem 4. *Floquet Theorem: The Mathieu equation has at least one solution $y(z)$ such that $y(z + \pi) = \sigma y(z)$ where σ is a constant.*

Theorem 5. *The product of the roots of the periodicity σ is unity.*

Corollary 1. *The Mathieu equation has at least one pseudo-periodic solution $y(z) = e^{\mu z}\phi(z)$ where μ is a constant and $\phi(z)$ has period π.*

Corollary 2. *If the Mathieu equation has a solution with periodicity factor $\sigma(\neq \pm 1)$ then $\exists$ an independent solution with periodicity factor σ^{-1}.*

Corollary 3. *$\sigma_1 = \sigma_2$ iff $\sigma_1 = \sigma_2 = \pm 1$.*

Corollary 4. *The Mathieu equation has a basically periodic solution, that is, periodic in π, iff the roots of the periodicity are equal.*

Corollary 5. *If $y_1(z)$ is a solution with periodicity factor σ, periodicity exponent μ, then $y_1(-z)$ is a solution with periodicity σ^{-1}, periodicity exponent $-\mu$.*

Theorem 6.

i) If $w_1(z)$ has period π, then the second solution has the form $w_2(z) = \pm\pi^{-1}w_2(\pi) \cdot z \cdot w_1(z) + u(z)$, where $u(z)$ has period π.

ii) If $w_2(z)$ has period π, then the second solution has the form $w_1(z) = \pm\pi^{-1}w_1'(\pi) \cdot z \cdot w_2(z) + u(z)$, where $u(z)$ has period π.

Summary

The nature of the general solution depends on the nature of the roots of the periodicity equation.

Case 1: The periodicity roots are different. $\sigma \neq \sigma^{-1}$. The periodicity roots have different periodicity exponents μ and $-\mu$. This results in two pseudo-periodic solutions:

$$\begin{aligned} y_1(z) &= e^{\mu z}\phi z \\ y_2(z) &= e^{-\mu z}\phi{-z} \end{aligned} \tag{21}$$

Case 2: The periodicity roots are identical. $\sigma_1 = \sigma_2 = \pm 1$. The periodicity roots have the same periodicity exponents, $\mu_1 = \mu_2$. This results in one basically periodic solution and one non-periodic, non-pseudo-periodic solution. for example:

$$\begin{aligned} y_1(z) &= \phi z \\ y_2(z) &= \pi w_2(\pi) \cdot z \cdot y_1(z) + u(z) \end{aligned}$$

5 Quiz Solutions [4]

Here are proofs for the necessary theorems and corrolaries for solving the Mathieu equation

$$\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} + (a - 2qcos(2t)) = 0. \tag{22}$$

Theorem 7. *If $r_1(t)$ is a solution, then so is $r_1(n\pi + t)$ for interger n.*

Proof. Let $t' = n\pi + t$.
Then $\mathrm{d}t'^2 = \mathrm{d}t^2$ and $\cos(2t') = \cos(2(n\pi + t)) = \cos(2n\pi)\cos(2t) - \sin(2n\pi)\sin(2t)$ $= \cos(2t)$. And thus we find that the Mathieu equation is invariant under the transformation $t \to t + n\pi$. □

Theorem 8. *∃ one odd and one even solution*

This follows from classical theory of differential equations.

Theorem 9.

i) $r_1(t)$ is even and $r_2(t)$ is odd

ii) $r_1(0) = \dot{r}_2(0) = 1; \dot{r}_1(0) = r_2(0) = 0$

iii) $r_1(t \pm \pi) = r_1(\pi)r_1(t) \pm \dot{r}_1(\pi)r_2(t)$

iv) $r_2(t \pm \pi) = \pm r_2(\pi)r_1(t) + \dot{r}_2(\pi)r_2(t)$

v) $r_1(t)\dot{r}_2(t) - \dot{r}_1(t)r_2(t) = 1$

vi) $r_1(\pi) = \dot{r}_2(\pi)$

Proof. i) This is equivalent to theorem 8.
ii) By an appropriate choice of the arbitrary constants $r_1(0)$ and $r_2(0)$ in agreement with i), the values of the derivatives also follow from the even/odd properties of the solutions.
iii) By theorem 7, if $r_1(t)$ is a solution, so is $r_1(t + \pi)$. Thus, this new solution is expressible in terms of the even and odd solutions. Taking the additive solution, we have
$r_1(t + \pi) = \alpha r_1(t) + \beta r_2(t)$ and $\dot{r}_1(t + \pi) = \alpha \dot{r}_1(t) + \beta \dot{r}_2(t)$. By setting $t = 0$, we find $\alpha = r_1(\pi)$ and $\beta = \dot{r}_1(\pi)$.
iv) Same principle of iii) apply.
v) Since the Mathieu equation has no terms in the first derivative, $\dot{r}$, it follows from the Abel identity that

$$r_1(t)\dot{r}_2(t) - r_2(t)\dot{r}_1(t) = Constant. \tag{23}$$

And since $r_1(0)\dot{r}_2(0) - r_2(0)\dot{r}_1(0) = 1$, by the choice of initial conditions it follows that

$$r_1(t)\dot{r}_2(t) - r_2(t)\dot{r}_1(t) = 1. \tag{24}$$

vi) By setting $t = \pi$ in the subtractive equations of iii) and iv) and using the periodicity given by theorem 7 with the results in ii), we find

$$r_1(0) = 1 = [r_1(\pi)]^2 - \dot{r}_1(\pi) r_2(\pi) \tag{25}$$
$$r_2(0) = 0 = r_2(\pi) r_1(\pi) - \dot{r}_2(\pi) r_2(\pi) \tag{26}$$
$$= r_2(\pi)[r_1(\pi) - \dot{r}_2(\pi)] \tag{27}$$

Then, either $r_1(\pi) - \dot{r}(\pi) = 0$ which is the result of interest, or $r_2(\pi) = 0$. For the latter, we find $[r_1(\pi)]^2 = 1$ and the expression in v) becomes $r_1(\pi)\dot{r}_2(\pi) = 1$. Equating the last two equations we find $r_1(\pi) = \dot{r}_2(\pi)$. □

Theorem 10. *Floquet Theorem: The Mathieu equation has at least one solution $y(t)$ such that $y(t+\pi) = \sigma y(t)$ where σ is a constant.*

Proof. : $y(t)$ can be expressed as a linear combination of the even and odd solution $r_1(t)$ and $r_2(t)$.
Let $y(t) = c \cdot r(t)$ where c is the row vector with arbitrary constant entries $\begin{pmatrix} c_1 & c_2 \end{pmatrix}$ and r(t) $= \begin{pmatrix} r_1(t) \\ r_2(t) \end{pmatrix}$. Then, from theorem 9 iii) and iv), we find,

$$y(t+\pi) = c \cdot r(t+\pi) = c \cdot A r(t) \tag{28}$$

where

$$A = \begin{pmatrix} r_1(\pi) & \dot{r}_1(\pi) \\ r_2(\pi) & \dot{r}_2(\pi) \end{pmatrix} \tag{29}$$

Thus to get solutions of the form $y(t+\pi) = \sigma y(t)$, we need

$$c \cdot A = \sigma c. \tag{30}$$

The solutions satisfy $|c \cdot (A - \sigma \cdot \mathbb{I}) \cdot c^{-1}| = 0$ or equivalently $|A - \sigma \cdot I| = 0$. Expanding the last expression yields

$$0 = (r_1(\pi) - \sigma)(\dot{r}_2(\pi) - \sigma) - \dot{r}_1(\pi) r_2(\pi) \tag{31}$$
$$= \sigma^2 - \sigma(r_1(\pi) + \dot{r}_2(\pi)) + r_1(\pi)\dot{r}_2(\pi) - \dot{r}_1(\pi) r_2(\pi) \tag{32}$$

The second term becomes $2\sigma r_1(\pi)$ by theorem 9 vi) and the last two terms equate to 1 by theorem 9 v). The periodicity equation becomes

$$\sigma^2 - 2\sigma r_1(\pi) + 1 = 0. \tag{33}$$

□

Theorem 11. *The product of the roots of the periodicity σ is unity.*

Proof. If we define σ_1 and σ_2 the roots of the periodicity equation obtained in the last theorem such that $(\sigma - \sigma_1)(\sigma - \sigma_2) = 0$. Thus, $\sigma_1 \sigma_2 = 1$. □

Corollary 6. *The Mathieu equation has at least one pseudo-periodic solution* $y(t) = e^{\mu t}\phi(t)$ *where we have defined* $\sigma = e^{\mu t}$ *with constant* μ *and* $\phi(t)$ *has period* π*.*

Proof.

$$\begin{aligned} y(t+\pi) &= \sigma y(t) \\ \phi(t+\pi) &= e^{-\mu(t+\pi)}y(t+\pi) \\ &= e^{-\mu\pi}e^{-\mu t}\sigma y(t) \\ &= \sigma^{-}1e^{-\mu t}\sigma y(t) \\ &= e^{-\mu t}y(t) = \phi(t) \end{aligned}$$

□

Corollary 7. *If the Mathieu equation has a solution with periodicity factor* $\sigma(\neq \pm 1)$ *then* $\exists$ *an independent solution with periodicity factor* σ^{-1}*.*

Proof. Let $y_1(t) = e^{\mu t}\phi(t)$ be a solution with periodicity σ such that $y_1(t+\pi) = \sigma y_1(t)$ and $\phi(t)$ has period π. By theorem 7, $y_2(t) \equiv y_1(-t) = e^{-\mu t}\phi(-t)$ is also a solution. We have

$$y_2(t+\pi) = y_1(-t-\pi) = e^{-\mu(-t-\pi)}\phi(-t-\pi) \tag{34}$$

$$= e^{-\mu\pi}e^{-\mu t}\phi(-t) \tag{35}$$

$$y_1(t+\pi) = e^{\mu(t+\pi)}\phi(t+\pi) \tag{36}$$

$$= e^{\mu t}e^{\mu\pi}\phi(t) \tag{37}$$

Taking their ratio, we find

$$\frac{y_2(t+\pi)}{y_1(t+\pi)} = e^{-2\mu\pi}\frac{e^{-\mu t}\phi(-t)}{e^{\mu t}\phi(t)} \tag{38}$$

$$= \sigma^{-2}\frac{y_2(t)}{y_1(t)} \tag{39}$$

$$\neq \frac{y_2(t)}{y_1(t)} \qquad \text{for } \sigma \neq \pm 1 \tag{40}$$

□

Corollary 8. $\sigma_1 = \sigma_2$ *iff* $\sigma_1 = \sigma_2 = \pm 1$.

This follows from $\sigma_1\sigma_2 = 1$ in theorem 5.

Corollary 9. *The Mathieu equation has a basically periodic solution, that is, periodic in* π*, iff the roots of the periodicity are equal.*

This follows from the expression $y(t+\pi) = \sigma y(t)$ and corollary 3.

Corollary 10. *If $y_1(t)$ is a solution with periodicity factor σ, periodicity exponent μ, then $y_1(-t)$ is a solution with periodicity σ^{-1}, periodicity exponent $-\mu$.*

This follows immediately form the proof of corollary 2.

Theorem 12.

i) If $r_1(t)$ has period π, then the second solution has the form $r_2(t) = \pm\pi^{-1} r_2(\pi) \cdot t \cdot r_1(t) + u(t)$, where $u(t)$ has period π.

ii) If $r_2(z)$ has period π, then the second solution has the form $r_1(z) = \pm\pi^{-1} \dot{r}_1(\pi) \cdot t \cdot r_2(t) + u(t)$, where $u(t)$ has period π.

[This is closely related to Ince's theorem, that the Mathieu equation never possesses 2 basically periodic solutions for the same values of a and q, expect for the trivial case of $q = 0$.]

Proof. For the first case i), we have from theorem 3 vi) combined with the periodicity of r_1

$$\dot{r}_2(\pi) = r_1(\pi) = r_1(0) = 1 \tag{41}$$

Then from theorem 3 iv), we have $r_2(t+\pi) = r_2(\pi)r_1(t) + r_2(t)$ which can be rearranged to

$$r_2(\pi)r_1(t) = r_2(t+\pi) - r_2(t) \tag{42}$$

Now let us define

$$u(t) = r_2(t) - \pi^{-1} r_2(\pi) \cdot t \cdot r_1(t). \tag{43}$$

Then

$$u(t+\pi) = r_2(t+\pi) - \pi^{-1} r_2(\pi) \cdot (t+\pi) \cdot r_1(t+\pi) \tag{44}$$

$$= r_2(t+\pi) - \pi^{-1} r_2(\pi) \cdot (t+\pi) \cdot r_1(t) \tag{45}$$

Thus

$$\begin{aligned} u(t+\pi) - u(t) &= r_2(t+\pi) - r_2(t) - \pi\pi^{-1} r_2(\pi) r_1(t) \\ &= r_2(\pi)r_1(t) - r_2(\pi)r_1(t) \qquad \text{by equation 42} \\ &= 0 \end{aligned}$$

Likewise, in case ii) where $r_2(t+\pi) = r_2(t)$, we find $r_1(\pi) = \dot{r}_2(\pi) = r_2(0) = 1$. Using theorem 3 iii), $r_1(t+\pi) = r_1(t) + \dot{r}_1(\pi r_2(t)$. We then define

$$u(t) = r_1(t) - \pi^{-1} \dot{r}_1(\pi) \cdot t \cdot r_2(t). \tag{46}$$

Then

$$u(t+\pi) = r_1(t+\pi) - \pi^{-1} \dot{r}_1(\pi)(t+\pi) r_2(t) \tag{47}$$

which can be used to show the required relation $u(t+\pi) = u(t)$ □

Summary

The nature of the general solution depends on the nature of the roots of the periodicity equation with two main cases to distinguish.

Case 1: The periodicity roots are different. $\sigma \neq \sigma^{-1}$. The periodicity roots have different periodicity exponents μ and $-\mu$. This results in two pseudo-periodic solutions:

$$y_1(t) = e^{\mu t}\phi(t) \tag{48}$$

$$y_2(t) = e^{-\mu t}\phi(-t) \tag{49}$$

Case 2: The periodicity roots are identical. $\sigma_1 = \sigma_2 = \pm 1$. The periodicity roots have the same periodicity exponents, $\mu_1 = \mu_2$. This results in one basically periodic solution and one non-periodic, non-pseudo-periodic solution. for example:

$$y_1(t) = \phi(t) \tag{50}$$

$$y_2(t) = \pi y_2(\pi) \cdot t \cdot y_1(t) + u(t) \tag{51}$$

[Recall from theorem 5 that the product of the roots of the periodicity factor is unity. And in corollary 1 we have made the definition $\sigma = e^{\mu t}$. Thus

$$\sigma_1 \sigma_2 = e^{(\mu_1 + \mu_2)\pi} = 1 \tag{52}$$

Hence $\mu_1 + \mu_2 = 2ni$ with integer n. It follows that in Case 2, $\mu_1 = \mu_2 = integer$.

6 Other Requirements

6.1 Resonator

One of the necessary requirements for the successful storing of ions in a radio-frequency trap is the production of a single RF with minimal noise. Typical trapping conditions for a Paul trap are a drive frequency between $\omega \simeq 10$MHz and $\omega \simeq 50$MHz and a trapping voltage amplitude between $V \simeq 200V$ and $V \simeq 400V$. A helical resonator can be introduced to set up an LCR system which, at the resonance frequency, can be used to supply a highly amplified single rf voltage to the trapping electrodes. Impedance matching nullifies the power reflected at the load hence maximising the power at the trapping electrodes whilst reducing noise due to (multiple) reflections. The higher the quality factor of the resonator, the larger the step-up in voltage. To measure the quality factor, the helical resonator can be connected to the trap and a tracking generator used with a directional coupler to obtain a spectrum of the output voltage against the drive frequency. [6] gives a detailed account on how to build a suitable helical resonator.

6.2 Vacuum

An ultra high vacuum is necessary in most, if not all, ion trap experiments. One reason for this is to ensure stably trapped ions by minimising the collision frequency with other elements. Another important reason is to avoid dark ions (mainly through charge exchange collisions). Dark ions are other species that have been affected and trapped by the system. They are so called because they do not fluoresce under the same lasers as they have different transition frequencies. Dark ions can be very common when not working in a 'good' vacuum environment. They are easily identified by the presence of a vacant space in a chain of ions, for example.

Officially, ultra high vacuum (UHV) is in the range 10^{-7} - 10^{-14} mbar. However, ion traps are typically ran at 10^{-10} mbar or below. To achieve these pressures and maintain them a series of pumps are used. In the ITCM group at Sussex, a roughing pump is first used to get pressures down to 10^{-2} mbar. A turbomolecular pump that is backed by this pressure is then used to bring the trap to roughly 10^{-8} mbar. The trap is then baked (insulated and heated to about $100°C$ to rid the trap walls of any condensation) for about a week. Following the cooling period the pressure reaches 10^{-10} mbar and the system can be sealed. Adsorption pumps and sublimations pumps are used to further reduced and maintain the low pressures. [7] is a comprehensice and excellent guide to vacuum technology.

6.3 Review Paper

Here's another piece I consider essential to ion trapping: the review paper *Quantum dynamics of single trapped ions* [8]. Unless you are trapping ions for studies in cold chemistry, you will want to do some quantum state engineering. This review paper covers the essentials including two-level systems and laser cooling.

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